Problems for Fall 2023

PROBLEMS

Proposals

To be considered for publication, solutions should be received by November 1, 2023.

2171. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

Evaluate the following sums in closed form.

(a) $$\sum_{n=0}^{\infty}\left(\cos x - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + (-1)^{n-1}\frac{x^{2n}}{(2n)!}\right)$$

(b) $$\sum_{n=0}^{\infty}\left(\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots (-1)^{n-1}\frac{x^{2n+1}}{(2n+1)!}\right)$$

2172. *Proposed by George Stoica, Saint John, NB, Canada.*

Let $A_1, \ldots, A_n$ be arbitrary events in a probability field. Denote by B_k the event that at least k of $A_1, \ldots, A_n$ occur. Prove that $\sum_{k=1}^{n} P(B_k) = \sum_{k=1}^{n} P(A_k)$.

2173. *Proposed by Quang Hung Tran, Hanoi, Vietnam.*

Let $ABCD$ and $XYZW$ be two squares with opposite orientations (for example, if $ABCD$ is oriented clockwise, $XYZW$ is oriented counterclockwise). Suppose that the perpendicular bisectors of the segments AB and XW meet at P and those of segments AD and XY meet at Q. Suppose that line AC meets line XZ at M and line BD meets line YW at N. Prove that $PMQN$ is a square.

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We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at `http://mathematicsmagazine.submittable.com`. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* `mathmagproblems@maa.org`.

2174. *Proposed by Ioan Băetu, Botoşani, Romania.*

The dihedral group

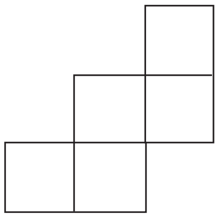
$$G = D_{2n} = \langle r, s | r^n = s^2 = e, srs^{-1} = r^{-1} \rangle$$

and the subgroup $H = \langle r \rangle$ give an example of a group G and a subgroup H such that $x^2 = e$ for all $x \in G - H$.

Show that if H is a subgroup of G, $x^2 = e$ for all $x \in G - H$, and $|G| > 2|H|$, then G is abelian.

2175. *Proposed by Jacob Siehler, Gustavus Adolphus College, Saint Peter, MN.*

For which integers $n \geq 3$ can the $n \times n$ square grid be colored black and white (using both colors at least once) so that every possible placement of a W pentomino (shown in the figure) on the grid covers an even number of black squares? The pentomino may be placed in any orientation.



PI MU EPSILON: PROBLEMS AND SOLUTIONS: SPRING 2023

STEVEN J. MILLER (EDITOR)

1. Problems: Spring 2023

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

Solutions and new problems should be emailed to the Problem Section Editor Steven J. Miller at sjm1@williams.edu; proposers of new problems are strongly encouraged to use LaTeX. Please submit each proposal and solution preferably typed or clearly written on a separate sheet, properly identified with your name, affiliation, email address, and if it is a solution clearly state the problem number. Solutions to open problems from any year are welcome, and will be published or acknowledged in the next available issue; if multiple correct solutions are received the first correct solution will be published (if the solution is not in LaTeX, we are happy to work with you to convert your work). Thus there is no deadline to submit, and anything that arrives before the issue goes to press will be acknowledged. Starting with the Fall 2017 issue the problem session concludes with a discussion on problem solving techniques for the math GRE subject test.

Earlier we introduced changes starting with the Fall 2016 problems to encourage greater participation and collaboration. First, you may notice the number of problems in an issue has increased. Second, any school that submits correct solutions to at least two problems from the current issue will be entered in a lottery to win a pizza party (value up to $100). Each correct solution must have at least one undergraduate participating in solving the problem; if your school solves $N \geq 2$ problems correctly your school will be entered $N \geq 2$ times in the lottery. Solutions for problems in the Spring Issue must be received by October 31, while solutions for the Fall Issue must arrive by March 31 (though slightly later may be possible due to when the final version goes to press, submitting by these dates will ensure full consideration). The winning school from the Fall problem set is **Christopher Newport University**.

#1394: *Proposed by Steven Miller, Williams College.* When we dropped my son off at Camp Winadu we received a blue and white half moon (those are his camp's colors). A half moon is a delicious frosted cookie, with frosting of one color on half of the circle and another color on the other. Splitting it in two is easy to do so that everyone gets an equal amount of the two colors. What is a good way to split it into three equal parts where each has the same amount of each color? Describe exactly where you make the cuts, assume all you can do is

Date: July 21, 2023.

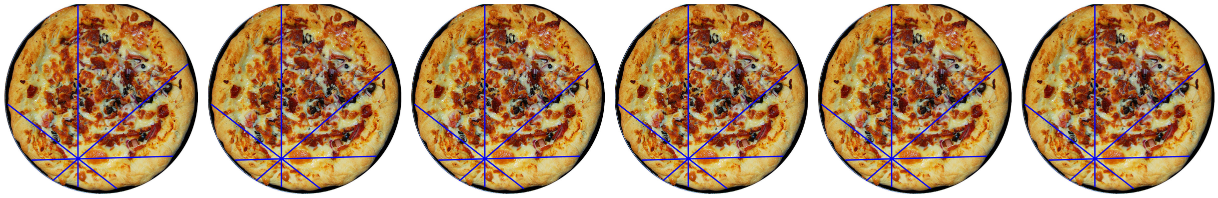


FIGURE 1. Pizza motivation; can you name the theorem that's represented here?

cut in any straight line. (Note of course it is trivial if you do not care about the amount of each color; choose 3 equi-spaced points on the perimeter and cut from the center to these.)

#1395: *Proposed by Hongwei Chen, Christopher Newport University.* Show that the Fourier sine series of $\ln(\tan x)$ on $(0, \pi/2)$ is given by

$$\ln(\tan x) \;=\; -\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{h_n}{n}\sin(4nx), \tag{1}$$

where

$$h_n \;=\; 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}.$$

Motivation: It is well-known that $\{\cos(2nx)\}_{n=0}^{\infty}$ forms an orthogonal basis of $L^2(0, \pi/2)$. In particular, we have

$$\begin{aligned} -\ln(\sin x) &= \ln 2 + \sum_{k=1}^{\infty} \frac{1}{k}\cos(2kx); \\ -\ln(\cos x) &= \ln 2 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\cos(2kx). \end{aligned}$$

This yields the Fourier cosine series of $\ln(\tan x)$:

$$-\ln(\tan x) \;=\; 2\sum_{k=0}^{\infty} \frac{1}{2k+1}\cos(2(2k+1)x), \qquad x \in (0, \pi/2). \tag{2}$$

It is natural to ask for its corresponding Fourier sine series. Since

$$h_n \;=\; H_{2n} - \frac{1}{2}H_n,$$

where H_n is the nth harmonic number, our formula enables us to find some exact values of Euler sums. For example, applying Parseval's identity, together with

$$\int_0^{\pi/2} \ln^2(\tan x)\,dx \;=\; \frac{1}{8}\pi^3,$$

we recover the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2}h_n^2 \;=\; \frac{1}{32}\pi^4.$$

#1396: *Proposed by Suaib Lateef, Oke oore, Iwo, Osun State, Nigeria.*

Let

$$f_{(n,p)} = \sum_{k=0}^{n} \binom{n}{k}^{p},$$

where $\binom{n}{k}$ is the standard binomial coefficient $\frac{n!}{k!(n-k)!}$. In 1894, Franel[1] showed that

$$(n+1)^2 f_{(n+1,3)} = (7n^2+7n+2) f_{(n,3)} + 8n^2 f_{(n-1,3)},$$

which can be re-written as

$$f_{(n,3)} = \frac{(7n^2-7n+2) f_{(n-1,3)} + 8(n-1)^2 f_{(n-2,3)}}{n^2}, \tag{1}$$

for $n > 1$. Also in 1895, Franel[2] proved that

$$(n+1)^3 f_{(n+1,4)} = 2(2n+1)(3n^2+3n+1) f_{(n,4)} + 4(4n-1)(4n+1) f_{(n-1,4)},$$

which can also be re-written as

$$f_{(n,4)} = \frac{2(2n-1)(3n^2-3n+1) f_{(n-1,4)} + 4(4n-5)(4n-3) f_{(n-2,4)}}{n^3}, \tag{2}$$

for $n > 1$.

V. Strehl[3] in1994 showed that

$$f_{(n,3)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}. \tag{3}$$

We can see that the right-hand sides of (1) and (3) are two different expressions for $f_{(n,3)}$. One could be curious to know if there are many other expressions for $f_{(n,3)}$ and possibly $f_{(n,p)}$, for all real and complex p. This curiosity leads us to ask the following question:
If p is any real or complex number, n is any positive integer, $\binom{n}{k}$ is a Binomial coefficient, and $\pm a_2$, $\pm a_3$, $\pm a_4 ... \pm a_r$ are some integers, does

$$\sum_{k=0}^{n} \binom{n}{k}^{p} \left(1 + \sum_{j=2}^{r} \pm a_j \left(\frac{k}{n}\right)^{j}\right) = 0$$

exist for all $r \geq 2$?

As a start, prove

$$\sum_{k=0}^{n} \binom{n}{k}^3 = 6 \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n-1}{k}^2 - 4 \sum_{k=0}^{n-1} \binom{n-1}{k}^3.$$

[1] J. Franel, *On a question of Laisant*, L'intermiaire des Mathaticiens, **1** (1894), 45–47.
[2] J. Franel, *On a question of Laisant*, L'intermiaire des Mathaticiens, **2** (1895), 33–35.
[3] V. Strehl, *Binomial Identities-combinatorial and algorithmic aspects*, Discrete Math., **136** (1994), 309–346.

#1397: *Proposed by Suaib Lateef, Oke oore, Iwo, Osun State, Nigeria.* Prove

$$\sum_{k=0}^{n}\binom{n}{k}^{p}\left(1-6\left(\frac{k}{n}\right)^{2}+4\left(\frac{k}{n}\right)^{3}\right) \ = \ 0$$

and

$$\sum_{k=0}^{n}\binom{n}{k}^{p}\left(1-4\left(\frac{k}{n}\right)^{2}-4\left(\frac{k}{n}\right)^{3}+10\left(\frac{k}{n}\right)^{4}-4\left(\frac{k}{n}\right)^{5}\right) \ = \ 0.$$

#1398: *Proposed by Carsten Botts (Johns Hopkins University, Applied Physics Lab) and Steven J. Miller (Williams College).* Let $p(x)$ be a continuous probability distribution (so it is non-negative and integrates to 1) such that the logarithm of $p(x)$ is three times continuously differentiable. Construct such a function with infinitely many points of inflection. *Note: some people use inflection point to mean a point where the second derivative vanishes, while others use it to be a point where the function changes from concave to convex; if the third derivative is non-zero at the point then these two definitions are equivalent.*

#1399: *Proposed by Zhongxue Lü (Jiangsu Normal University) and Steven J. Miller (Williams College).* This problem is inspired from an observation in 2021 (due to the backlog of problems it is only being published now), where 2021 is formed by writing two consecutive integers one after the other; in other words it is of form $n \cdot 10^{k} + (n+1)$ where k is the number of digits of n and n has leading digit non-zero and is not all 9's. We call such integers 2-adjacent joined numbers. Note we do not consider 102 or 10000 such numbers (even though the first could be written as $01 * 10^2 + 02$ and the second as $99 * 10^2 + 100$). How many 2-adjacent joined numbers are there less than 10^{100}?

GRE Practice #11:

Let $f(x) = \sum_{n=1}^{\infty} x^n/n$ for $-1 < x < 1$. Then $f'(x)$ equals
(a) $\frac{1}{1-x}$ (b) $\frac{x}{1-x}$ (c) $\frac{1}{1+x}$ (d) $\frac{x}{1+x}$ (e) 0.

PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be uploaded to the submission management system ***Submittable*** by visiting the web address https://cmj.submittable.com/submit (instructions are provided at this site). Alternatively, problem proposals may be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be uploaded to the submission management system ***Submittable*** by visiting the web address https://cmj.submittable.com/submit (instructions are provided at this site). Alternatively, solutions may be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than September 15, 2023. Sending both pdf and TEX files is ideal.

PROBLEMS

1246. *Proposed by Cezar Lupu, Yanqi Lake BIMSA and Tsinghua University, Beijing, China.*

Let $f : [0, 1] \to \mathbb{R}$ be an integrable function. Now set

$$\mu_k := \int_0^1 x^k f(x)dx \text{ for } k = 0, 1, 2.$$

Prove that $\frac{1}{3}\int_0^1 f^2(x)dx \geq 10\mu_2(\mu_0 + \mu_1) - \mu_1(8\mu_0 + 9\mu_1)$

1247. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

For every positive integer n, set

$$u_n = \sum_{k=1}^{n} \frac{1}{n + \sqrt{nk}}.$$

Find $\lim_{n\to\infty} \left(\frac{u_n}{2-\ln(4)}\right)^n$.

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This article has been corrected with minor changes. These changes do not impact the academic content of the article.

1248. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let R be a commutative ring with identity. For the purposes of this problem, say that R is *special* provided R satisfies the following conditions:

1. R has at least two distinct proper, nonzero ideals, and
2. if I and J are distinct, proper, nonzero ideals of R, then $I + J := \{i + j : i \in I, j \in J\} \notin \{I, J\}$.

Find all special rings (up to isomorphism).

1249. *Proposed by Didier Pinchon, Toulouse, France, and George Stoica, Saint John, Canada.*

Let $a_1, a_2, \ldots, a_n$ be real numbers such that

$$\sum_{1\le i_1<i_2<\cdots<i_k\le n} (1-a_{i_1})(1-a_{i_2})\cdots(1-a_{i_k}) = 0 \text{ for } k = 1, 2, \ldots, n.$$

Prove that $\sum_{1\le i_1<i_2<\cdots<i_k\le n} a_{i_1}\cdot a_{i_2}\cdots a_{i_k} = \binom{n}{k}$ for $k = 1, 2, \ldots, n$.

1250. *Proposed by Jeff Stuart, Pacific Lutheran University, Tacoma, WA.*

Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of complex numbers, and let b and c also be complex numbers. For every positive integer k, define the $k \times k$ matrix A_k by

$$A_k := \begin{bmatrix} a_1 & b & b & b & b & \cdots & b \\ c & a_2 & b & b & b & \cdots & b \\ c & c & a_3 & b & b & \cdots & b \\ c & c & c & a_4 & b & \cdots & b \\ c & c & c & c & a_5 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & b \\ c & c & c & c & \cdots & c & a_k \end{bmatrix}.$$

1. When all $a_j = a$ for some complex number a and when $b = c$, establish the following equation without using the formula in 2. or the recursion in 3.: for $k \ge 1$, $\det(A_k) = (a-b)^{k-1}(a + kb - b)$.
2. When $k \ge 2$ and $b = c$, show that $\det(A_k) = \left(1 - b\frac{d}{db}\right)\prod_{j=1}^{k}(a_j - b)$.
3. Show that for $k \ge 2$, we have $\det(A_{k+1}) = (a_k + a_{k+1} - b - c)\det(A_k) - (b - a_k)(c - a_k)\det(A_{k-1})$.

PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Hongwei Chen, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed problems must not be under consideration concurrently at any other journal, nor should they be posted to the internet before the deadline date for solutions. Proposed solutions to the problems below must be submitted by December 31, 2023. Proposed classics should include the problem statement, solution, and references. More detailed instructions are available online. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12405. *Proposed by George Stoica, Saint John, NB, Canada.* Let q be an odd prime, and let F_q be the field with q elements. Define permutations ρ, σ, and τ of F_q by $\rho(x) = x + 1$, $\sigma(x) = x^{q-2}$, and $\tau(x) = -x^{q-2}$.
(a) Prove that ρ and σ generate the full symmetric group S_q if $q \equiv 1 \pmod 4$ and the alternating group A_q if $q \equiv 3 \pmod 4$.
(b) Prove that the permutations ρ and τ generate the full symmetric group S_q for all q.

12406. *Proposed by Raymond Mortini, University of Luxembourg, Esch-sur-Alzette, Luxembourg, and Rudolf Rupp, Nuremberg Institute of Technology, Nuremberg, Germany.* For fixed $p \in \mathbb{R}$, find all functions $f : [0, 1] \to \mathbb{R}$ that are continuous at 0 and 1 and satisfy $f\left(x^2\right) + 2pf(x) = (x + p)^2$ for all $x \in [0, 1]$.

12407. *Proposed by an anonymous contributor, New Delhi, India.* Let r be a positive real number. Evaluate

$$\int_0^\infty \frac{x^{r-1}}{\left(1+x^2\right)\left(1+x^{2r}\right)}\,dx.$$

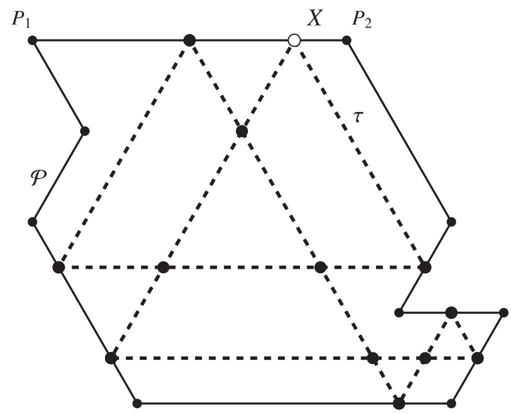
12408. *Proposed by Tran Quang Hung, Hanoi, Vietnam.* Let $ABCD$ be a trapezoid with AB parallel to CD. Let EBC and FAD be similar isosceles triangles with $EB = EC$ and $FA = FD$ erected externally to $ABCD$. Let P be the point such that EP is perpendicular to DB and FP is perpendicular to AC. Prove $PA = PB$.

12409. *Proposed by Erik Vigren, Swedish Institute of Space Physics, Uppsala, Sweden.* Let n be a positive integer, let v_0 be the zero vector in $\{0, 1\}^n$, and choose $v_1 \in \{0, 1\}^n$. Define vectors $v_k \in \{0, 1\}^n$ as follows. For $k \geq 2$, work modulo 2 and let $v_k = v_{k-1} + v^*_{k-1} + v^*_{k-2}$, where $(x_1, \ldots, x_n)^* = (x_2, \ldots, x_n, x_1)$. Write Σv for the sum of the entries of vector v.
(a) Show that $\sum_{k=1}^{2n} \Sigma v_k \leq n^2$.
(b) For which choices of v_1 does equality hold in **(a)**?

doi.org/10.1080/00029890.2023.2210053

12410. *Proposed by Tigran Hakobyan, Yerevan State University, Vanadzor, Armenia.* Which sets S of positive integers are such that the product of any number of elements from S, allowing repetitions, has the form a^b for integers a and b with $a \geq 1$ and $b \geq 2$?

12411. *Proposed by Colin Defant, Princeton University, Princeton, NJ, and Pakawut Jiradilok, Massachusetts Institute of Technology, Cambridge, MA.* Let $\mathcal{P} = P_1 P_2 \ldots P_n$ be a simple polygon that is a union of triangles chosen from the infinite plane tiling by equilateral triangles of side length 1. Let X be a point on $P_1 P_2$ such that $X P_1$ is not an integer. Assume the sides of $\mathcal{P}$ are mirrors and let τ be the path of a light ray that emerges from X at an angle of 60° to $P_1 P_2$.



(a) Prove that τ returns to X.

(b) The points where τ intersects either itself or $\mathcal{P}$ divide τ into segments. In the diagram, τ has 18 segments. Prove that the number of segments is divisible by 3.