Suggested Talks and Problems

Talk Topics for 2022-2023 School Year

Below are some possible topics. I hope these include something for each taste. (I can also make other suggestions.)

Mathematics Education

1. Geometric Solutions of Cubic Equations - The 11th century Persian poet, mathematician, and astronomer Omar Khayyam devised a geometrical method to also solve the cubic equation $ax^3 + bx^2 + cx + d = 0$. There are many references for this problem.
2. Elementary Methods for Solving Calculus Problems - Many calculus problems can be solved using algebra and geometry, without using the mathematical tools developed in calculus. The Armenian astrophysicist Mamikon A. Mnatsakanian went to Caltech after the fall of Soviet Union and returned to his first love of developing geometric and visual methods for solving calculus problems. There he worked with the famous American mathematician Tom M. Apostol at "Project MATHEMATICS!". Any of their papers include a wealth of insight and are great for presentations at any level. The first paper was titled "A Visual Approach to Calculus Problems" which you can find below. References: http://calteches.library.caltech.edu/4007/1/Calculus.pdf, http://www.its.caltech.edu/~mamikon/calculus.html and http://www.projectmathematics.com/.
3. Short biographical talks on female mathematicians, for example, Margherita Piazzolla Beloch, Elizabeth Smith, etc. See https://en.wikipedia.org/wiki/List_of_women_in_mathematics.
4. Present the paper *An Alternative to Integration by Partial Fractions Technique*. This is the title of a short paper by Yusuf Gurtas published in The College Mathematics Journal, vol. 50 (2019), no. 2, 140-142, and available at https://www.tandfonline.com/doi/full/10.1080/07468342.2019.1561125.
5. Present the paper *A Simple Proof of Descartes' Rule of Sign*. This is the title of a short paper by Xiaoshen Wang published in Amer. Math. Monthly, vol. 111 (2004), no. 4, 525-526, and available at https://www.tandfonline.com/doi/abs/10.1080/00029890.2004.11920108.

Applied Mathematics

6. Presentation of the calculus based proof of $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$. You learned in Calculus II that this infinite series is convergent (p-series with $p = 2 > 1$). In Boundary Value Problems, you use Fourier series to find its value. But this can be done elegantly using calculus and is included in the book *Proofs from THE BOOK* by Martin Aigner and Günte Ziegler, available in the library. However, there are other proofs. The paper http://math.cmu.edu/~bwsulliv/basel-problem.pdf lists several solutions with references.

7. With COVID-19 epidemic, the interest in mathematical models of diseases is on the rise. They span statistical, discrete, and continuous models. For example, discrete or continuous SIR or SEIR models. Any overview talk of such models or specific development of models for COVID-19 will make an interesting talk.
8. Fibonacci Day is Nov 23 (1123). This year a talk on difference equations, which leads to Fibonacci sequence formula, will be very informative. A good source is the textbook An Introduction to Difference Equations, by Saber Elaydi, ISBN: 978-0387230597, available in the library.

Mathematics

9. Almost Perfect Numbers - For a natural number n, call sum of its proper divisors s(n). A natural number is perfect if it is sum of its proper divisors: s(n)=n. For example, 6 is a perfect number since 1+2+3=6. A natural number is called almost perfect (also called perfect-minus-one) if sum of its proper divisors is one less than the number: s(n)=n-1. For example, 8 is an almost perfect number since 1+2+4=8-1. The only known almost perfect numbers are of powers of 2. The only known odd almost perfect number is $2^0 = 1$. The research project is to quantify, as much as possible, almost perfect numbers. This investigation can include both theoretical and computational work. References: http://www.jstor.org/stable/2689036, http://www.jstor.org/stable/2303162 and http://www.ams.org/journals/mcom/1981-36-154/S0025-5718-1981-0606516-3/.
10. Pi Day (Mar 14, 3/14). A talk on number π; history, calculation, properties, and applications. Last year we had a talk on its irrationality, a talk on pi being transcendental will complete proof of these important properties.
11. Proof of Convergence of Fourier Series. Read, understand, and rewrite the proof in your own words. Reference: David Powers, BVP's and PDE's, 6th edition, ISBN 978-0-12-374719-8.
12. Present the paper *Finding Real Roots of Polynomials Using Sturm Sequences*. This article compares Descartes' Rule of Signs, the Budan-Fourier Theorem, and versions of Sturm's Method in contrast with the approximate root count gleaned from graphing utilities. This articles is published at PRIMUS, (30)1:36-49, 2020, and also available at https://www.tandfonline.com/doi/full/10.1080/10511970.2018.1501626.
13. Present the paper *Inflating the Cube Without Stretching*. This is the title of a short paper by Igor Pak published in Amer. Math. Monthly, vol. 115 (2008), no. 5, 443-445, and available at http://www.math.ucla.edu/~pak/papers/milka2.pdf. In this article the author describes a way to deform a cube that distances between points are maintained while the volume is increased. The beauty of this article is that one can actually construct the deformed cube!
14. Address the conjectures in the paper https://maa.tandfonline.com/doi/pdf/10.1080/0025570X.2020.1704613 regarding certain primes. One such prime is 29 for which $2(29) + 9 = 67$ is also a prime and again $6(67) + 7 = 409$ is another prime.

15. The Arithmetic Mean – Geometric Mean (AM-GM) Inequality has many proofs and applications. A new proof is at https://maa.tandfonline.com/doi/pdf/10.1080/07468342.2020.1697605. An overview of this topic and presentation of interesting proofs will make a nice talk.

Statistics/Data Science

16. Give a talk on probability of randomly selected positive integers being relatively prime. A quick search will result in many references.

17. Present the paper Absent-Minded Passengers published in American Mathematical Monthly, 126:10,867-875, and available at https://www.tandfonline.com/doi/full/10.1080/00029890.2019.1656024. Here is the abstract of the paper. Passengers board a fully booked airplane in order. The first passenger picks one of the seats at random. Each subsequent passenger takes his or her assigned seat if available, otherwise takes one of the remaining seats at random. It is well known that the last passenger obtains her own seat with probability 1/2. We study the distribution of the number of incorrectly seated passengers, and we also discuss the case of several absent-minded passengers.

18. Present the paper Three Persons, Two Cuts: A New Cake-Cutting Algorithm in Mathematics Magazine, Vol 95, Issue 2, 110-122, and available at https://maa.tandfonline.com/doi/full/10.1080/0025570X.2022.2023300

Problems for Spring 2023

PROBLEMS

Proposals

To be considered for publication, solutions should be received by March 1, 2023.

2151. *Proposed by Tran Quang Hung, Hanoi, Vietnam.*

Let $ABCD$ and $XYZT$ be two directly similar squares such that A and Y lie on the lines XT and CD, respectively. Let M be the intersection of lines XZ and AC, and let N be the intersection of lines XY and BC. Prove that circumcenter of $\triangle XAC$ lies on the line MN.

2152. *Proposed by Paul Bracken, University of Texas Rio Grande Valley, Edinburg, TX.*

Evaluate

$$\int_0^1 \int_0^1 \frac{dy\, dx}{\sqrt{1-x^2}\sqrt{1-y^2}\,(1+xy)}.$$

2153. *Proposed by Rex H. Wu, New York, NY.*

Let F_n and L_n be the Fibonacci and Lucas numbers, respectively. Evaluate the following for $k \geq 0$.

(a) $\displaystyle\sum_{n=0}^{\infty} \arctan \frac{F_{2k}}{F_{2n+1}}$

(b) $\displaystyle\sum_{n=0}^{\infty} \arctan \frac{L_{2k+1}}{L_{2n}}$

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We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this Magazine.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com*. More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by* LaTeX *source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

2154. *Proposed by the Columbus State University Problem Solving Group, Columbus, GA.*

Let $f(n)$ denote the number of ordered partitions of a positive integer n such that all of the parts are odd. For example, $f(5) = 5$, since 5 can be written as $5, 3+1+1, 1+3+1, 3+1+1$, and $1+1+1+1+1$. Determine $f(n)$.

2155. *Proposed by Ioan Băetu, Botoşani, Romania.*

Let R be a ring with identity and U a subset of the units of R with $|U| = p$, where p is an odd prime. Suppose that for all $a \in R$, there is a $u \in U$ and a $k \in \mathbb{Z}^+$ such that $ua^k = a^{k+1}$. Show that

(a) For all $a \in R$, there is a $u \in U$ such that $ua = a^2$.

(b) The ring R is commutative.

PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than March 15, 2023. Sending both pdf and TEX files is ideal.

PROBLEMS

1231. *Proposed by George Apostolopoulos, Messolonghi, Greece.*

Let ABC be a triangle. Show that $\sum_{\alpha=\angle A,\angle B,\angle C} \sin^3(\alpha)\cos(\alpha) \le \frac{9\sqrt{3}}{16}$.

1232. *Proposed by Jacob Guerra, Salem State University, Salem, MA.*

Define, for every non-negative integer n, the nth Catalan number by $C_n := \frac{1}{n+1}\binom{2n}{n}$. Consider the sequence of complex polynomials in z defined by $z_k := z_{k-1}^2 + z$ for every non-negative integer k, where $z_0 := z$. It is clear that z_k has degree 2^k and thus has the representation $z_k = \sum_{n=1}^{2^k} M_{n,k} z^n$, where each $M_{n,k}$ is a positive integer. Prove that $M_{n,k} = C_{n-1}$ for $1 \le n \le k+1$.

1233. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Suppose that X and Y are independent, uniform random variables over [0, 1]. Define U_X, V_X, and B_X as follows: U_X is uniform over $[0, X]$, V_X is uniform over $[X, 1]$, and $B_X \in \{0, 1\}$, with $P(B_X = 1) = X$, and $P(B_X) = 0 = 1 - X$. Now define random variables Z and W_X as follows:

$$Z = (Y - X)\mathbf{1}\{Y \ge X\} + (1 - X + Y)\mathbf{1}\{Y < X\}, \text{ and}$$

$$W_X = B_X \cdot U_X + (1 - B_X)V_X.$$

Prove that both Z and W_X are uniform over [0, 1]. Here, $\mathbf{1}[S]$ is the indicator function that is equal to 1 if S is true and 0 otherwise.

doi.org/10.1080/07468342.2022.2100155

1234. *Proposed by Moubinool Omarjee, Lycée Henry IV, Paris, France.*

For every positive integer n, set $a_n := \sum_{k=1}^{n} \frac{1}{k^4}$ and $b_n := \sum_{k=1}^{n} \frac{1}{(2k-1)^4}$. Compute $\lim_{n\to\infty} n^3 (\frac{b_n}{a_n} - \frac{15}{16})$.

1235. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let S be a set, and let $f: S \to S$ be a function. For $s \in S$, the *orbit* of s is defined by $\mathcal{O}(s) := \{f^n(s): n \geq 0\}$, where $f^0: S \to S$ is the identity map and f^n is the n-fold composition of f with itself for $n > 0$. A subset $X \subseteq S$ is *closed under* f provided that for all $x \in X$, also $f(x) \in X$. Finally, if X is closed under F, we say that X is *finitely generated* if there is a finite $F \subseteq X$ such that $X = \bigcup_{x\in F} \mathcal{O}(x)$. Find all structures (S, f) up to isomorphism where S is not finitely generated, but every proper subset of S closed under f is finitely generated. Note that (S, f) and (T, g) are *isomorphic* if there is a bijection $\varphi: S \to T$ such that $\varphi(f(s)) = g(\varphi(s))$ for all $s \in S$.

PROBLEMS AND SOLUTIONS

Edited by **Daniel H. Ullman, Daniel J. Velleman, Stan Wagon, and Douglas B. West**

with the collaboration of Paul Bracken, Ezra A. Brown, Hongwei Chen, Zachary Franco, George Gilbert, László Lipták, Rick Luttmann, Hosam Mahmoud, Frank B. Miles, Lenhard Ng, Rajesh Pereira, Kenneth Stolarsky, Richard Stong, Lawrence Washington, and Li Zhou.

Proposed problems, solutions, and classics should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed problems must not be under consideration concurrently at any other journal, nor should they be posted to the internet before the deadline date for solutions. Proposed solutions to the problems below must be submitted by April 30, 2023. Proposed classics should include the problem statement, solution, and references. More detailed instructions are available online. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12356. *Proposed by Ira Gessel, Brandeis University, Waltham, MA.* Let $A(z) = z^3 - z^2$ and $B(z) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\binom{3n+1}{n} z^{n+1}$. Prove that B is a one-sided inverse to A in the sense that $A(B(z)) = z$. Also, prove $B(A(z)) = 1 - z^2 M(-z)$, where

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

(The coefficients of $M(z)$ are the Motzkin numbers $1, 1, 2, 4, 9, 21, \ldots$.)

12357. *Proposed by Van Khea, Prey Veng, Cambodia, and Dan Ştefan Marinescu, Hunedoara, Romania.* Suppose that triangles ABC and DEF have the same centroid, where D, E, and F are on the segments BC, CA, and AB, respectively. Let I be the incenter of triangle ABC. Prove

$$\frac{AI}{AD} + \frac{BI}{BE} + \frac{CI}{CF} \le 2.$$

12358. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* For a positive integer q and a set A of positive integers, say that A is q-good if every sufficiently large integer has exactly q representations as the sum of distinct elements of A.
(a) Which sets A are 1-good?
(b) For which q does there exist a q-good set?
(c) For q as in **(b)**, which sets A are q-good?

12359. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.* Let n be a positive integer. Prove

$$\frac{-1-\pi}{4n} - \frac{1}{8n^2} < \sum_{k=1}^{n} \frac{1}{(2k-1)^2} - 2\left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1}\right)^2 < \frac{-1+\pi}{4n} - \frac{1}{8n^2}.$$

doi.org/10.1080/00029890.2022.2120717

12360. *Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania.* Evaluate

$$\lim_{n\to\infty} \frac{(n+1)^2}{x_{n+1}} - \frac{n^2}{x_n},$$

where $x_n = \sqrt[n]{\sqrt{2!}\sqrt[3]{3!}\cdots\sqrt[n]{n!}}$.

12361. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* For a nonengative integer k, let $r_{3k}=0$, $r_{3k+1}=1$, and $r_{3k+2}=-1$. Prove

$$\sum_{k=0}^{n-1}\binom{2k}{k} = \sum_{k=0}^{n} r_k \binom{2n}{n-k},$$

for every positive integer n.

12362. *Proposed by Antonio Garcia, Strasbourg, France.* Evaluate

$$\lim_{n\to\infty}\int_0^{\pi/2} \frac{n}{\left(\sqrt{2}\cos x\right)^n + \left(\sqrt{2}\sin x\right)^n}\,dx.$$

PROBLEMS

Proposals

To be considered for publication, solutions should be received by May 1, 2023.

2156. *Proposed by Cezar Lupu, Tsinghua University, Beijing, China.*

Let $ABCD$ be a convex quadrilateral in the plane with vertices having rational coordinates. Let P be a point in its interior having rational coordinates such that

$$m\angle PAB = m\angle PBC = m\angle PCD = m\angle PDA = q\pi, \text{ with } q \in \mathbb{Q}.$$

Show that $ABCD$ is a square. Give an example to show that the condition that $q \in \mathbb{Q}$ cannot be dropped.

2157. *Proposed by Philippe Fondanaiche, Paris, France.*

Consider two sequences. One is the number of digits in the base 2 representation of 10^k, $k = 1, 2, \ldots$, and the other is the number of digits in the base 5 representation of 10^k, $k = 1, 2, \ldots$. Show that every integer greater than 1 appears in exactly one of the two sequences. Which sequence contains 2023?

2158. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

(a) Arrange the integers from 1 to 15 (inclusive) in a row so that the sum of any two adjacent numbers is a perfect square.

(b) Find the smallest positive integer n such that the integers from 1 to n can be arranged in a circle so that the sum of any two adjacent numbers is a perfect square. Justify your answer.

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2159. *Proposed by George Stoica, Saint John, NB, Canada.*

Two players, A and B, alternately throw a pair of dice with A going first. Let $a, b \in \{2, 3, \ldots, 12\}$ be fixed. Player A wins by having a roll worth a points before player B has a roll worth b points. Otherwise, player B wins.

What is the probability that player A wins?

2160. *Proposed by Gregory Dresden, Washington & Lee University, Lexington, VA.*

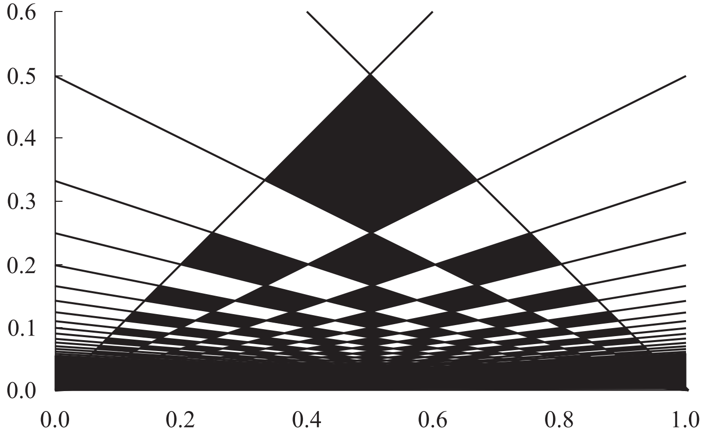
Consider the lines

$$y = x/1, \ y = x/2, \ y = x/3, \ y = x/4, \ldots$$

and the lines

$$y = (1-x)/1, \ y = (1-x)/2, \ y = (1-x)/3, \ y = (1-x)/4, \ldots,$$

which intersect to form an infinite number of quadrilaterals. Starting with the lozenge at the top, shade every other quadrilateral, as shown in the figure.



Find the total area of all the shaded quadrilaterals.

PROBLEMS AND SOLUTIONS

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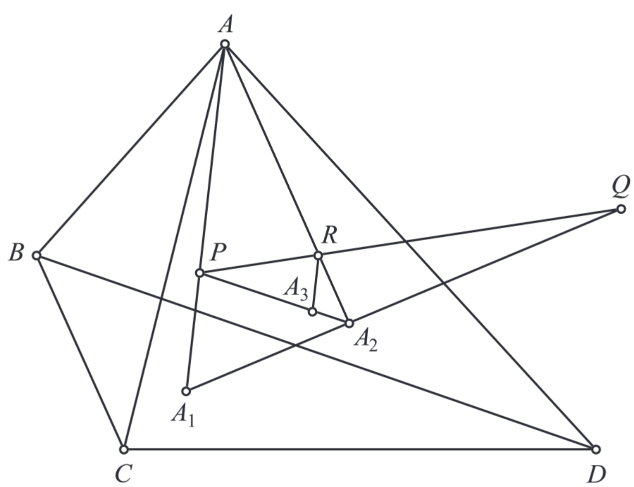
PROBLEMS

1236. *Proposed by Tran Quang Hung, High School for Gifted Students, Vietnam National University, Hanoi, Vietnam.*

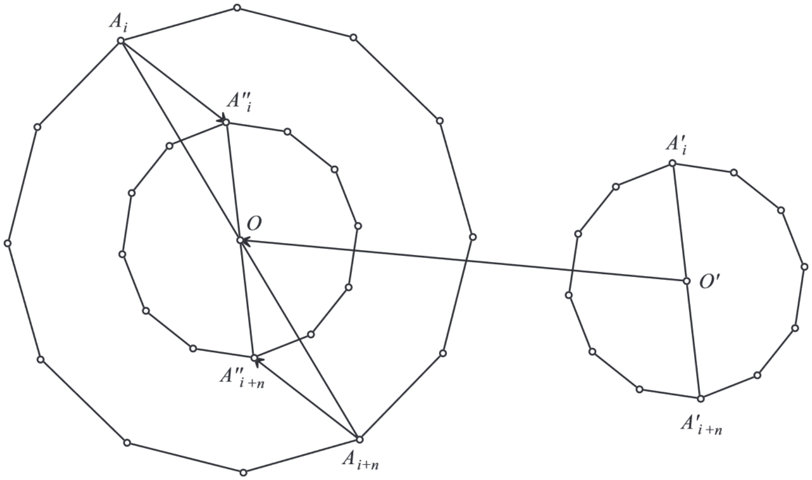
Let $ABCD$ be a tetrahedron in 3-space, and let P, Q and R be three collinear points. Assume that lines PA, PB, PC and PD are not parallel to planes (BCD), (CDA), (DAB) and (ABC), respectively. Line PA meets plane (BCD) at point A_1. In the plane (APR), assume that the two lines AR and A_1Q intersect at A_2. Point A_3 lies on line PA_2 such that RA_3 is parallel to line AA_1. Define similarly the points B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2 and D_3. Prove that R is the centroid of tetrahedron $A_3B_3C_3D_3$ (see figure).



doi.org/10.1080/07468342.2022.2127287

1237. *Proposed by Tran Quang Hung, High School for Gifted Students, Vietnam National University, Hanoi, Vietnam.*

Let $A_1A_2 \ldots A_{2n}$ and $A'_1A'_2 \ldots A'_{2n}$ $(n \geq 2)$ be two directly $2n$-regular polygons. Prove that $\sum_{i=1}^{n} A_{2i}A'^2_{2i} = \sum_{i=0}^{n-1} A_{2i+1}A'^2_{2i+1}$ (see figure).



1238. *Proposed by Jacob Siehler, Gustavus Adolphus College, St. Peter, MN.*

Consider the intersection of a unit square with a copy of itself rotated through an angle of θ about their mutual center. Note that in general, this region is an octagon. Evaluate the average area of the intersection as θ ranges from 0 to $\frac{\pi}{2}$.

1239. *Proposed by Moubinool Omarjee, Lycée Henri IV, Paris, France.*

Let u_0 be a positive real number, and for every $n \in \mathbb{N}$, define $u_{n+1} := \frac{u_n^3 - 3u_n - \sqrt{5}}{3u_n^2 + 3\sqrt{5}u_n + 4}$. Find a closed-form expression for u_n in terms of u_0 and n.

1240. *Proposed by Eden Ketchum (student) and Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let S be a set. Recall that a *partial order* on S is a binary relation $\leq$ which is reflexive, anti-symmetric, and transitive. If S, T are sets and $\leq, \preceq$ are partial orders on S and T, respectively, then we say that the partially ordered sets $(S, \leq)$ and $(T, \preceq)$ are *isomorphic* if there is a bijection $f: S \to T$ such that for all $s_1, s_2 \in S$: $s_1 \leq s_2$ iff $f(s_1) \preceq f(s_2)$. Now let F be a field, and let $\mathcal{P}^+(F)$ be the collection of additive subgroups of F, partially ordered by set-theoretic inclusion, and let $\mathcal{P}^\times(F)$ be the collection of multiplicative subgroups of $F^\times := F\backslash\{0\}$, partially ordered by inclusion. Find all fields F for which $\mathcal{P}^+(F)$ and $\mathcal{P}^\times(F)$ are isomorphic.